Complexity of Social Welfare Optimization in Multiagent Resource Allocation

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ABSTRACT

We study the complexity of social welfare optimization in multi-agent resource allocation. We assume resources to be indivisible and nonshareable and agents to express their utilities over bundles of resources, where utilities can be represented in either the bundle form or the *k*-additive form. Solving some of the open problems raised by Chevaleyre et al. [2] and confirming their conjectures, we prove that egalitarian social welfare optimization is NP-complete for both the bundle and the 1-additive form, and both exact utilitarian and exact egalitarian social welfare optimization are DP-complete, each for both the bundle and the 2-additive form, where DP is the second level of the boolean hierarchy over NP. In addition, we prove that social welfare optimization with respect to the Nash product is NP-complete for both the bundle and the 1-additive form. Finally, we briefly discuss hardness of social welfare optimization in terms of inapproximability.

Categories and Subject Descriptors

F.2 [Theory of Computation]: Analysis of Algorithms and Problem Complexity;
I.2.11 [Artificial Intelligence]: Distributed Artificial Intelligence—*Multiagent Systems*;
J.4 [Computer Applications]: Social and Behavioral Sciences—*Economics*

General Terms

Economics, Theory

Keywords

Multiagent resource allocation, Social welfare optimization, Computational complexity, Economically-motivated agents, Auction and mechanism design

1. INTRODUCTION

In multiagent resource allocation (see, e.g., [14, 5, 7, 4, 6, 1] and the survey [2]), autonomous agents (e.g., bidders in an auction) express their utilities over bundles of resources, where we assume resources to be indivisible and nonshareable. The aim is to obtain an allocation of these bundles of resources to the agents. A particularly important task in this regard is social welfare optimization. An allocation procedure assigns bundles of resources to the agents, who thus may realize their individual utilities for the bundles received. A system designer, however, typically takes a more global perspective and seeks to optimize the allocation procedure not toward increasing the individual utility of a single agent, but rather toward maximizing social welfare of the whole "society of agents." The most central concepts of social welfare are utilitarian and egalitarian social welfare, and a compromise of both ideas is the Nash product (see, e.g., Chevaleyre et al. [2]).

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In a bit more detail, utilitarian social welfare sums up the agents' individual utilities in a given allocation, thus providing a useful measure of the overall—and also of the average—benefit for society. For instance, in a combinatorial auction the auctioneer's aim is to maximize the sum of the prizes paid for the items auctioned, no matter which agent can realize which utility.

In contrast, egalitarian social welfare gives the utility of the agent who is worst off in a given allocation, which provides a useful measure of fairness in cases where the minimum needs of all agents are to be satisfied. For example, think of distributing humanitarian aid items (such as food, medical aid, blankets, tents, etc.) among the needy population in a disaster area (e.g., an area hit by an earthquake or a tsunami). Guaranteeing every survivor's continuing survival is the primary goal in such a scenario, and it is best captured by the notion of egalitarian social welfare.

The Nash product is the product of the agents' individual utilities and can be thought of as combining utilitarian and egalitarian social welfare. On the one hand, just as utilitarian social welfare, the Nash product increases even with single increasing individual utilities that can be realized by the agents. On the other hand, just as egalitarian social welfare, the Nash product reaches its maximum when the utilities realized are distributed equally over all the agents, whereas social welfare in terms of the Nash product vanishes as soon as there is only one agent realizing no utility at all (and it sharply decreases when there are agents whose realized utility drops down to a value close to zero).

We study the computational complexity of social welfare optimization problems. A crucial aspect here is how the agents' utility functions are represented. The most basic representation forms are the bundle form and the *k*-additive form, see, e.g., Chevaleyre et al. [2]. The bundle form (which, in terms of combinatorial auctions, corresponds to the XOR bidding language, see, e.g., [2]) simply enumerates all bundles (with a nonzero utility) and attaches a numerical utility value to each bundle. While this form is fully expressive, it is not very compact in general, i.e., its size can be exponential in the number of resources. The *k*-additive representation form can be more succinct if *k* is small enough; however, it is fully expressive only for sufficiently large *k* (see [4]).

The complexity of utilitarian and egalitarian social welfare optimization with respect to these forms of representing utilities has been investigated by various authors (see, e.g., [7, 4, 6, 1]). Chevaleyre et al.'s comprehensive survey of issues in multiagent resource allocation [2] presents a number of complexity results obtained, and also a number of conjectures regarding open issues. We solve in the aff rmative four of the seven open complexity assertions conjectured in [2] by proving that egalitarian social welfare optimization is NP-complete for both the bundle and the 1-additive form, and both exact utilitarian and exact egalitarian social welfare optimization are DP-complete, each for both the bundle and the 2-additive form, where DP is the second level of the boolean hierarchy over NP (see [12]; some motivation for studying completeness in DP is given in Section 2). In addition, we prove that social welfare optimization with respect to the Nash product is NP-complete for both the bundle and the 1-additive form. Finally, we brief y discuss some of the hardness results for social welfare optimization in terms of inapproximability.

Organization of the Paper.

The general framework in which our problems are formalized is described in Section 2. Section 3 presents previous results and related work and gives an overview of our results. Section 4 provides our complexity results for the bundle form and Section 5 those for the k-additive form. Section 6 gives a short discussion of inapproximability for social welfare optimization, and Section 7, f nally, concludes by stating some open questions.

2. PRELIMINARIES AND NOTATION

Basic Notions from Multiagent Resource Allocation.

We adopt the framework for multiagent resource allocation described in [2]. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of n agents, let $R = \{r_1, r_2, \ldots, r_m\}$ be a set of m indivisible and nonshareable resources (i.e., each resource is assigned as a whole and can be assigned to only one agent), and let $U = \{u_1, u_2, \ldots, u_n\}$ be a set of utility functions. Letting 2^R denote the set of subsets of R, $u_i : 2^R \to \mathbb{Q}$ gives agent a_i's utility for each bundle of resources (independently of the utilities of other agents).

An *allocation for A and R* is a mapping $X : A \to 2^R$ with $X(a_j) \cap X(a_k) = \emptyset$ for any two agents a_j and a_k, $j \neq k$. As a shorthand, we write $u_i(X)$ for $u_i(X(a_i))$, i.e., for the utility agent a_i can realize in allocation X. The set of all possible allocations for A and R is denoted by $\Pi_{n,m}$ and has cardinality $\|\Pi_{n,m}\| = n^m$. We'll refer to such a triple (A, R, U) as a *multiagent resource allocation setting* (or a *MARA setting*, for short).

Utility functions can be given in different ways, and the representation form potentially affects the complexity of the problems we will consider later. We will focus on the following two representation forms (see [2]):

1. The *bundle form*: Agent a_i's utility for any bundle $R' \subseteq R$ of resources is given by $(R', u_i(R'))$, where pairs with $u_i(R') = 0$ are omitted. This representation is "fully expressive" (i.e., every utility function can be described in bundle form); however, the size of descriptions can be exponential in the number of resources.

2. The *k-additive form*, for a f xed positive integer k: Agent a_i's utility for any bundle $R' \subseteq R$ of resources is given as

$$u_i(R') = \sum_{T \subseteq R', \|T\| \leq k} \alpha_i^T,$$

where for each bundle $T \subseteq R'$ with $\|T\| \leq k$, α_i^T is a unique coeff cient expressing the "synergetic" value of agent a_i owning all the resources in T. This representation form is fully expressive only if k is large enough. Choosing k to be relatively small allows for a relatively succinct representation of utility functions. The k-additive form was proposed for representing utilities in multiagent resource allocation by Chevaleyre et al. [3, 4] and, independently, in combinatorial auctions by Conitzer et al. [5].

Alternatively, *straight-line programs* (i.e., the *SLP form*) could be used for representing utilities, see [2, 7]. As we won't consider this form, we don't def ne it here.

To measure the quality of an allocation, various notions of social welfare have been proposed (see, e.g., [2]). In this paper, we focus on the following types of social welfare.

DEFINITION 2.1. *For* (A, R, U) *a MARA setting and* X *an allocation for A and R,*

1. *the* utilitarian social welfare induced by X *is defined as*

$$sw_u(X) = \sum_{a_j \in A} u_j(X),$$

2. *the* egalitarian social welfare induced by X *is defined as*

$$sw_e(X) = \min\{u_j(X) \mid a_j \in A\},$$

3. *the* Nash product induced by X *is defined as*

$$sw_N(X) = \prod_{a_j \in A} u_j(X).$$

Chevaleyre et al. [2] considered various social welfare optimization problems. We use the following naming scheme, where the subscript $\texttt{form} \in \{\texttt{bundle}, \texttt{1-additive}, \texttt{2-additive}, \ldots\}$ represents either one of the above representation forms for utility functions (i.e., the bundle form or the k-additive form for $k \in \{1, 2, \ldots\}$). The following social welfare optimization problems will be of central importance in this paper.

UTILITARIAN SOCIAL WELFARE OPTIMIZATION$_{\texttt{form}}$	
Given:	A MARA setting (A, R, U), where $\|A\| = \|U\| = n$ and $\|R\| = m$ and where $\texttt{form}$ indicates how the utility functions in U are represented, and $K \in \mathbb{Q}$.
Question:	Does there exist an allocation $X \in \Pi_{n,m}$ such that $sw_u(X) \geq K$?

We use the shorthand USWO$_{\texttt{form}}$ to denote the above problem. EGALITARIAN SOCIAL WELFARE OPTIMIZATION$_{\texttt{form}}$ and NASH PRODUCT SOCIAL WELFARE OPTIMIZATION$_{\texttt{form}}$ (which we abbreviate as ESWO$_{\texttt{form}}$ and NPSWO$_{\texttt{form}}$, respectively) are def ned by changing "$sw_u(X)$" into, respectively, "$sw_e(X)$" and "$sw_N(X)$" in the question f eld. The exact variant of, e.g., USWO$_{\texttt{form}}$ is denoted by XUSWO$_{\texttt{form}}$ and is def ned as follows.

EXACT UTILITARIAN SOCIAL WELFARE OPTIMIZATION$_{\texttt{form}}$	
Given:	A MARA setting (A, R, U), where $\|A\| = \|U\| = n$ and $\|R\| = m$ and where $\texttt{form}$ indicates how the utility functions in U are represented, and $K \in \mathbb{Q}$.
Question:	Does it hold that $\max\{sw_u(X) \mid X \in \Pi_{n,m}\} = K$?

Changing "$sw_u(X)$" into, respectively, "$sw_e(X)$" and "$sw_N(X)$" in the question f eld, we obtain EXACT EGALITARIAN SOCIAL WELFARE OPTIMIZATION$_{\texttt{form}}$ (XESWO$_{\texttt{form}}$, for short) and EXACT NASH PRODUCT SOCIAL WELFARE OPTIMIZATION$_{\texttt{form}}$ (XNPSWO$_{\texttt{form}}$, for short).

In addition, Chevaleyre et al. [2] considered the problem of whether there exists an envy-free allocation. Envy-freeness means that every agent is at least as satisfied with his or her bundle of resources in a given allocation as he or she would be with any other agent's bundle. Formally speaking, an allocation X for A and R is said to be *envy-free* if for any two agents a_i and a_j in A, we have $u_i(X(a_i)) \geq u_i(X(a_j))$. ENVY-FREENESS$_{\texttt{form}}$ (EF$_{\texttt{form}}$, for short) denotes the problem of determining, given a MARA setting (A,R,U), whether there exists an envy-free allocation $X \in \Pi_{n,m}$.

Basic Notions from Complexity Theory.

We assume the reader is familiar with the basic complexity-theoretic notions such as the complexity classes P (deterministic polynomial time), NP (nondeterministic polynomial time), and coNP, the notion of (polynomial-time many-one) reducibility, denoted by $\leq_{\rm m}^{\rm p}$, and the standard notions of hardness for and completeness in a complexity class $\mathscr{C}$ (with respect to $\leq_{\rm m}^{\rm p}$).

Papadimitriou and Yannakakis [12] introduced the complexity class DP as the set of differences of any two NP sets, i.e., $\mathrm{DP} = \{A - B \mid A, B \in \mathrm{NP}\}$. DP is known to be the second level of the boolean hierarchy over NP and captures the complexity of the *exact variants of many* NP*-hard optimization problems*. For example, suppose the optimum value $\max\{sw_u(X) \mid X \in \Pi_{n,m}\}$ of utilitarian social welfare in a given MARA setting equals 2315. Then a "yes" instance of USWO$_{\texttt{form}}$, say $(A,R,U,17)$, merely indicates that this maximum is at least 17, whereas $(A,R,U,17)$ would be a "no" instance of XUSWO$_{\texttt{form}}$ and the only "yes" instance $(A,R,U,2315)$ of XUSWO$_{\texttt{form}}$ provides a much more precise information. Needless to say that deciding whether such an optimum is hit exactly is a computationally more challenging task than deciding whether it falls into some range: DP is widely believed to be a strictly more powerful class than NP. Other natural problems that typically are complete for DP are *unique solution problems*, which test the uniqueness of solutions for NP problems, and *critical graph problems* where a small change of the input graph (such as adding just one edge or deleting just one vertex and its incident edges) switches some property of this graph (e.g., its three-colorability). For more background on computational complexity (especially with regard to DP and the boolean hierarchy over NP), we refer to [12] and the textbook [13].

3. RELATED WORK AND OVERVIEW OF OUR RESULTS

Bouveret and Lang [1] proved, among many other results, that EF$_{\texttt{bundle}}$ is NP-complete. Chevaleyre et al. [4] proved that both USWO$_{\texttt{bundle}}$ and USWO$_{\texttt{2-additive}}$ are NP-complete. When utilities are represented in the SLP form, NP-completeness is also known for the problem of deciding whether there is an envy-free allocation for a given MARA setting [6] and for the utilitarian social welfare optimization problem [7]. Regarding the remaining cases, Chevaleyre et al. [2] conjectured that for each of the bundle, the 2-additive, and the SLP form, (1) egalitarian social welfare optimization is NP-complete, and (2) exact utilitarian social welfare optimization is DP-complete. We prove these two conjectures in the affirmative for the bundle form in Section 4, and for the k-additive form in Section 5. In addition, for both the bundle and the k-additive form, we prove that exact egalitarian social welfare optimization is DP-complete and that social welfare optimization via the Nash product is NP-complete.

Chevaleyre et al. [2] also conjectured that EF$_{\texttt{2-additive}}$ is NP-complete. However, an affirmative solution to this conjecture follows from a result of Lipton et al. [11] who proved that EF$_{\texttt{1-additive}}$ is NP-complete. Note that EF$_{\texttt{1-additive}}$ is a special case of EF$_{\texttt{2-additive}}$, so the latter problem immediately inherits the NP-hardness lower bound of the former problem; membership of EF$_{\texttt{2-additive}}$ in NP is easy to see.

4. COMPLEXITY OF SOCIAL WELFARE OPTIMIZATION: BUNDLE FORM

As mentioned above, Chevaleyre et al. [2] conjectured that ESWO$_{\texttt{bundle}}$ is NP-complete. Our first result solves this conjecture in the affirmative, and in addition shows the same result for NPSWO$_{\texttt{bundle}}$.

THEOREM 4.1. *ESWO$_{\texttt{bundle}}$ and NPSWO$_{\texttt{bundle}}$ are NP-complete.*

Proof. Membership in NP is easy to see for both problems: Given an instance (A,R,U,K), where (A,R,U) is a MARA setting and $K \in \mathbb{Q}$, in polynomial time we can nondeterministically guess an allocation and then deterministically compute the minimum (respectively, the product) of the agents' utilities and compare it with K.

We show both hardness results via a single reduction from 3-SAT, one of the standard NP-complete problems (see, e.g., Garey and Johnson [8]), which is defined as follows: Given a boolean formula φ in conjunctive normal form with at most three literals per clause, is there a truth assignment to the variables of φ that makes φ evaluate to true?

We are given an instance φ of 3-SAT. Let $C = \{c_1, c_2, \ldots, c_n\}$ be the set of clauses of φ. Without loss of generality, we assume φ to contain only variables that occur both as a positive literal and as a negative one. If we have a variable that does not occur in both ways, the clause containing this variable can always be satisfied, and so deleting such a clause does not affect the formula's satisfiability (i.e., φ is satisfiable if and only if the thus reduced formula is satisfiable). Furthermore, we assume that there are at least two clauses (i.e., $n \geq 2$) and no clause contains any variable twice (be it as a positive or as a negative literal).

We introduce one agent a_j for each clause c_j of φ and an additional agent a_0, resulting in a set $A = \{a_0, a_1, \ldots, a_n\}$ of agents. Depending on the literals of the clauses, we define the set R of resources as follows. We introduce a new resource for each occurrence of a literal in a clause and each occurrence of the negation of the same literal in a clause to the right. In more detail, we define a resource for each pair $((\ell,s),(\neg\ell,t))$ exactly if either ℓ or $\neg\ell$ is a variable of φ and $1 \leq s < t \leq n$, where (k,i) indicates the occurrence of literal k in clause c_i.

Now, we set the utilities of the agents a_j in A, $j \neq 0$, as follows. Agent a_j's nonzero utilities depend on the clause c_j.[1] For each literal ℓ or $\neg\ell$ in c_j, agent a_j forms a bundle with all pairs $((\ell,s),(\neg\ell,t))$ where either $s = j$ or $t = j$, and assigns utility one. If the negated literal $\neg\ell$ occurs only once, the corresponding bundle contains only a single resource. Furthermore, if the clause contains at least two literals then agent a_j assigns a utility of one to each combination of two of these bundles, and, analogously, if the clause contains three literals then agent a_j assigns a utility of one to the combination of all three bundles. Since each clause contains at most three literals, each agent assigns nonzero utilities to at most seven nonempty bundles. Finally, agent a_0 has a utility of one for the empty bundle and a utility of $n = \|C\|$ for the bundle

[1] Regarding an empty bundle of resources, each agent a_j, $j \neq 0$, assigns utility value zero.

containing all resources.[2]

In addition, we choose the same parameter $K = K_e = K_N = 1$ for our instances of ESWO$_{\texttt{bundle}}$ and NPSWO$_{\texttt{bundle}}$, namely, (A,R,U,K_e) and (A,R,U,K_N). It is easy to see that (A,R,U,K) can be computed in polynomial time from φ, since each clause consists of at most three variables and thus each agent forms utilities for at most seven nonempty bundles.

Note that each truth assignment to the variables of φ corresponds to an assignment of the resources in R to the agents a_j, $1 \le j \le n$, as follows. If ℓ is a literal in clause c_j that is true under a given truth assignment, then agent a_j is assigned the bundle consisting of all resources $((\ell,j),(\neg\ell,t))$ with $j<t$ and $((\neg\ell,s),(\ell,j))$ with $s<j$. If none of the literals in clause c_j is true under a given truth assignment (i.e., clause c_j is evaluated to false and hence φ is not satisf ed), then agent a_j does not receive any resource.

We claim that there exists an allocation whose egalitarian social welfare is exactly $K=1$ if and only if φ is satisf able.

From left to right, suppose there exists an allocation X with $sw_e(X)=1$. So a_0 is assigned the empty bundle with utility one. If all resources are allocated according to X then a truth assignment to the variables of φ that makes φ true can be obtained as follows. If agent a_j, $1\le j\le n$, is assigned resource $((\ell,s),(\neg\ell,t))$ with either $s=j$ or $t=j$ then literal ℓ can be set so as to satisfy clause c_j. Thus, if agent a_j holds at least one nonempty bundle, clause c_j is satisf ed. Note that the assignment of pairwise disjoint bundles does not allow to assign the same value to both literal ℓ and its negation $\neg\ell$. Since $sw_e(X)=1$, it follows from the def nition of egalitarian social welfare that even the agent that among $a_1,\dots,a_n$ is worst off must hold a nonempty bundle. Thus, all clauses of φ are satisf ed under the truth assignment corresponding to X.

From right to left, if φ is satisf able, there is a truth assignment satisfying all clauses $c_j \in C$. In the corresponding allocation X of bundles of resources, each of the agents $a_1,\dots,a_n$ receives a nonempty bundle and so can realize a utility of one. Thus, a_0 is assigned the empty bundle, also with utility one. So $sw_e(X)=1$.

Analogously, one can show that there exists an allocation X for which the Nash product is exactly $K=1$ if and only if φ is satisf able. ❑

As pointed out by a reviewer, Theorem 4.1 can be proven by a simpler reduction from, e.g., EXACT SET COVER. However, the slightly more involved reduction from 3-SAT given above has a property[3] that will make Lemma 4.3 applicable and thus allows us to reuse and extend the above reduction to a reduction that works for proving Theorem 4.4 below.

The following example illustrates the reduction from the proof of Theorem 4.1 and will be continued in Example 4.5 to show how this reduction is modif ed for the proof of Theorem 4.4.

EXAMPLE 4.2. *Let*

$$\varphi \quad = \quad (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2) \wedge (\neg x_2 \vee \neg x_3)$$

be a given boolean formula. Note that φ is satisfiable, e.g., by the truth assignment $(1,0,0)$ to (x_1,x_2,x_3). According to the proof of Theorem 4.1, we introduce the agents a_0, a_1, a_2, and a_3 (since φ consists of $n=3$ clauses) and we have the following resources resulting from φ:

$$r_1 = ((x_1,1),(\neg x_1,2)), \quad r_2 = ((x_2,1),(\neg x_2,2)),$$
$$r_3 = ((x_2,1),(\neg x_2,3)), \quad r_4 = ((x_3,1),(\neg x_3,3)).$$

Since the first clause of φ contains three literals, agent a_1 has nonzero utilities for seven bundles of resources. However, a_2 and a_3 bid on only three bundles each, as the second and the third clause contain only two literals.[4] At last, agent a_0 always bids on two bundles (the empty one and the one with all resources), see Table 1.

Agent	Pairs (Bundle, Utility)
a_0	$(\emptyset,1)$, $(\{r_1,r_2,r_3,r_4\},3)$
a_1	$(\{r_1\},1)$, $(\{r_2,r_3\},1)$, $(\{r_4\},1)$, $(\{r_1,r_2,r_3\},1)$, $(\{r_1,r_4\},1)$, $(\{r_2,r_3,r_4\},1)$, $(\{r_1,r_2,r_3,r_4\},1)$
a_2	$(\{r_1\},1)$, $(\{r_2\},1)$, $(\{r_1,r_2\},1)$
a_3	$(\{r_3\},1)$, $(\{r_4\},1)$, $(\{r_3,r_4\},1)$

Table 1: Utilities of the agents resulting from φ

An allocation corresponding to the truth assignment $(1,0,0)$ for φ is obtained (according to the proof of Theorem 4.1) by giving $\emptyset$ to a_0, $\{r_1\}$ to a_1, $\{r_2\}$ to a_2, and $\{r_3,r_4\}$ to a_3. Thus, each agent can realize a utility of exactly one. This means that the utilitarian social welfare in this allocation X is $n+1=4$, whereas both the egalitarian social welfare of X and the Nash product of X are one.

As noted above, to apply Lemma 4.3 in the proof of Theorem 4.4 the maximum utilitarian social welfare in the MARA setting constructed from some unsatisfiable formula has to be the number of clauses (see Footnote 3). So consider the unsatisfiable formula

$$\psi \quad = \quad (y_1 \vee y_2) \wedge (\neg y_1 \vee y_2) \wedge (y_1 \vee \neg y_2) \wedge (\neg y_1 \vee \neg y_2)$$

with $n'=4$ clauses. Now, the reduction from the proof of Theorem 4.1 gives the following resources:

$$r_1 = ((y_1,1),(\neg y_1,2)), \quad r_2 = ((y_1,1),(\neg y_1,4)),$$
$$r_3 = ((y_2,1),(\neg y_2,3)), \quad r_4 = ((y_2,1),(\neg y_2,4)),$$
$$r_5 = ((\neg y_1,2),(y_1,3)), \quad r_6 = ((y_2,2),(\neg y_2,3)),$$
$$r_7 = ((y_2,2),(\neg y_2,4)), \quad r_8 = ((y_1,3),(\neg y_1,4)).$$

Table 2 shows the nonzero utilities of the agents a_0, $a_1,\dots,a_4$.

Agent	Pairs (Bundle, Utility)
a_0	$(\emptyset,1)$, $(\{r_1,r_2,r_3,r_4,r_5,r_6,r_7,r_8\},4)$
a_1	$(\{r_1,r_2\},1)$, $(\{r_3,r_4\},1)$, $(\{r_1,r_2,r_3,r_4\},1)$
a_2	$(\{r_1,r_5\},1)$, $(\{r_6,r_7\},1)$, $(\{r_1,r_5,r_6,r_7\},1)$
a_3	$(\{r_5,r_8\},1)$, $(\{r_3,r_6\},1)$, $(\{r_3,r_5,r_6,r_8\},1)$
a_4	$(\{r_2,r_8\},1)$, $(\{r_4,r_7\},1)$, $(\{r_2,r_4,r_7,r_8\},1)$

Table 2: Utilities of the agents resulting from ψ

Since the recources are indivisible and nonshareable, at least one of the agents a_i, $1\le i\le 4$, cannot realize any utility at all. Thus, in any allocation X, both the egalitarian social welfare of X and the Nash product of X is zero, and the utilitarian social welfare of X cannot be $n'+1=5$. However, all resources can be assigned to a_0, so the utilitarian social welfare in this allocation is $n'=4$.

[2] The utility for the bundle of all resources can be set to any positive integer value in this proof. However, in the upcoming proof of Theorem 4.4 (which reuses and extends the present construction), we need a_0 to have a utility of $n=\|C\|$ for this bundle.

[3] Namely, that the maximum utilitarian social welfare in the MARA setting constructed from formula φ over clause set C is exactly $\|C\|+1$ if φ is satisf able, and is exactly $\|C\|$ otherwise.

[4] When we say an agent *bids on a bundle*, we mean this agent assigns some nonzero utility to this bundle.

The above examples show that, as claimed in the proof of Theorem 4.1, the given formula is satisfiable if and only if there is some allocation whose egalitarian social welfare (respectively, whose Nash product) is exactly one.

Chevaleyre et al. [2] conjectured that $\text{XUSWO}_{\texttt{bundle}}$ is DP-complete. We solve this conjecture in the aff rmative, and we establish the corresponding result for $\text{XESWO}_{\texttt{bundle}}$. Theorem 4.4 below makes use of a result by Wagner [16], who provided suff cient conditions for proving hardness for each level of the boolean hierarchy over NP. We state Wagner's result here for the case of DP, the second level of this hierarchy.

LEMMA 4.3 (WAGNER [16]). *Let A be some* NP-*complete problem and let B be an arbitrary problem. If there exist a polynomial-time computable function f such that, for all input strings x_1 and x_2 for which $x_2 \in A$ implies $x_1 \in A$, we have that*

$$(x_1 \in A \wedge x_2 \notin A) \iff f(x_1,x_2) \in B, \qquad (4.a)$$

then B is DP-*hard.*

THEOREM 4.4. $\text{XUSWO}_{\texttt{bundle}}$ *and* $\text{XESWO}_{\texttt{bundle}}$ *are* DP-*complete.*

Proof. To prove membership of $\text{XUSWO}_{\texttt{bundle}}$ in DP, consider the condition $\max\{sw_u(X) \mid X \in \Pi_{n,m}\} = K$, where we may assume that $K \in \mathbb{Z}$.[5] Note that this condition is true if and only if

1. $(\exists X \in \Pi_{n,m})\ [sw_u(X) \geq K]$ and
2. $(\forall X \in \Pi_{n,m})\ [sw_u(X) < K+1]$.

Since the f rst condition is an NP predicate and the second condition is a coNP predicate, we can write $\text{XUSWO}_{\texttt{bundle}}$ as $C \cap \overline{D}$ for suitable NP sets C and D. Thus, $\text{XUSWO}_{\texttt{bundle}}$ is in DP.

The proof that $\text{XESWO}_{\texttt{bundle}}$ is in DP as well is analogous and thus omitted.

To show that $\text{XUSWO}_{\texttt{bundle}}$ is DP-hard, we apply Lemma 4.3 with $A = $ 3-SAT and $B = \text{XUSWO}_{\texttt{bundle}}$. Recall the construction presented in the proof of Theorem 4.1. Note that the maximum utilitarian social welfare is exactly $K = n+1$ if φ is satisf able (because each of the $n+1$ agents can realize a utility of exactly one in that case), and is $K = n$ otherwise (because either one agent a_i, $1 \leq i \leq n$, cannot realize any bundle at all, whereas the other agents a_j, $0 \leq j \leq n$ and $j \neq i$, will realize a utility of one each, or agent a_0 can realize a utility of n and all other agents cannot realize any utility); see also Example 4.2.

Let φ and ψ be two given boolean formulas in conjunctive normal form, where φ has n clauses and ψ has n' clauses and φ and ψ have disjoint variable sets. According to the hypothesis of Lemma 4.3, we assume that if ψ is satisf able then so is φ.

We apply the same construction as in the proof of Theorem 4.1 to both φ and ψ, thus obtaining two MARA settings, $(A^\varphi, R^\varphi, U^\varphi)$ and (A^ψ, R^ψ, U^ψ). Our construction is completed by merging them to obtain a MARA setting (A,R,U) with $A = A^\varphi \cup A^\psi$, $R = R^\varphi \cup R^\psi$, and $U = U^\varphi \cup U^\psi$, and by setting $K_{xu} = n + n' + 1$, so (A,R,U,K_{xu}) is our $\text{XUSWO}_{\texttt{bundle}}$ instance.

Since the variable sets of φ and ψ are disjoint, the sets of agents, A^φ and A^ψ, and the sets of resources, R^φ and R^ψ, are disjoint as well. Note further that each agent in A^φ bids only on bundles of resources from R^φ, and each agent in A^ψ bids only on bundles of resources from R^ψ.[6] From our construction and these observations it follows that:

1. If $\varphi \in$ 3-SAT and $\psi \in$ 3-SAT then there exists an allocation X with $sw_u(X) = n + n' + 2 > K_{xu}$, so
$$\max\{sw_u(X) \mid X \in \Pi_{\|A\|,\|R\|}\} > K_{xu}.$$
2. If $\varphi \in$ 3-SAT and $\psi \notin$ 3-SAT then there exists an allocation X with $sw_u(X) = n + n' + 1 = K_{xu}$ and there is no allocation Y with $sw_u(Y) > K_{xu}$, so
$$\max\{sw_u(X) \mid X \in \Pi_{\|A\|,\|R\|}\} = K_{xu}.$$
3. If $\varphi \notin$ 3-SAT and $\psi \notin$ 3-SAT then for any allocation X, $sw_u(X) \leq n + n' < K_{xu}$, so
$$\max\{sw_u(X) \mid X \in \Pi_{\|A\|,\|R\|}\} < K_{xu}.$$

The case that $\varphi \notin$ SAT and $\psi \in$ SAT cannot occur by our assumption that if ψ is satisf able then so is φ. Hence, ($\varphi \in$ 3-SAT and $\psi \notin$ 3-SAT) if and only if $(A,R,U,K_{xu}) \in \text{XUSWO}_{\texttt{bundle}}$, so (4.a) is true. By Lemma 4.3, $\text{XUSWO}_{\texttt{bundle}}$ is DP-hard.

To show that $\text{XESWO}_{\texttt{bundle}}$ is DP-hard, we again start from two formulas φ and ψ with disjoint variable sets and such that if ψ is satisf able then so is φ. We apply the same construction as above except with slightly different utilities. First, we double all utilities of the agents obtained from φ, so

- every agent a_i^φ, $1 \leq i \leq n$, now has a utility of two for each of the bundles mentioned in the proof of Theorem 4.1, and
- a_0^φ has a utility of two for the empty bundle and a utility of $2n$ for the bundle containing all resources obtained from φ.

Second, we adjust the utilities of the agents obtained from ψ. Again, we double all utilities, so

- every agent a_j^ψ, $1 \leq j \leq n'$, now has a utility of two for each of the bundles mentioned in the proof of Theorem 4.1, and
- a_0^ψ has a utility of two for the empty bundle and a utility of $2n'$ for the bundle containing all resources obtained from ψ.

In addition, each agent a_j^ψ, $1 \leq j \leq n'$, has a utility of one for the empty bundle. This means that each agent a_j^ψ can realize a utility of one even if a_j^ψ doesn't get any resource.

Merging the MARA settings $(A^\varphi, R^\varphi, U^\varphi)$ and (A^ψ, R^ψ, U^ψ) resulting from φ and ψ, respectively, we obtain a MARA setting (A,R,U) as above, and we set $K_{xe} = 1$, so (A,R,U,K_{xe}) is our $\text{XESWO}_{\texttt{bundle}}$ instance. It follows that:

1. If $\varphi \in$ 3-SAT and $\psi \in$ 3-SAT then there is an allocation X with $sw_e(X) = 2 > K_{xe}$, so
$$\max\{sw_e(X) \mid X \in \Pi_{\|A\|,\|R\|}\} > K_{xe}.$$
2. If $\varphi \in$ 3-SAT and $\psi \notin$ 3-SAT then there is an allocation X with $sw_e(X) = 1 = K_{xe}$ and there is no allocation Y with $sw_e(Y) > K_{xe}$, so
$$\max\{sw_e(X) \mid X \in \Pi_{\|A\|,\|R\|}\} = K_{xe}.$$
3. If $\varphi \notin$ 3-SAT and $\psi \notin$ 3-SAT then for any allocation X, $sw_e(X) = 0 < K_{xe}$, so
$$\max\{sw_e(X) \mid X \in \Pi_{\|A\|,\|R\|}\} < K_{xe}.$$

[5] This assumption can be made without loss of generality, because we can multiply all utilities and K by their least common multiple.

[6] This implies that no agent in A bids on bundles of resources from both R^φ and R^ψ.

Again, the case that $\varphi \notin$ SAT and $\psi \in$ SAT cannot occur. Hence, ($\varphi \in$ 3-SAT and $\psi \notin$ 3-SAT) if and only if $(A,R,U,K_{xe}) \in$ XESWO$_{\texttt{bundle}}$, so (4.a) is true. By Lemma 4.3, XESWO$_{\texttt{bundle}}$ is DP-hard. ❑

EXAMPLE 4.5 (CONTINUING EXAMPLE 4.2). *Considering the two boolean formulas φ and ψ from Example 4.2, we see that*

$$(\varphi \in \text{3-SAT} \textit{ and } \psi \notin \text{3-SAT}) \implies (A,R,U,K_{xu}) \in \text{XUSWO}_{\texttt{bundle}}, \quad (4.b)$$

where $K_{xu} = n + n' + 1 = 3 + 4 + 1 = 8$ and (A,R,U) is the MARA setting constructed from φ (with three clauses) and ψ (with four clauses) as in the proof of Theorem 4.4. Note that (4.b) corresponds, for the concrete formulas φ and ψ from Example 4.2, to the second case in establishing (4.a) to show XUSWO$_{\texttt{bundle}}$ DP-*hard via Lemma 4.3 in this proof.*

5. COMPLEXITY OF SOCIAL WELFARE OPTIMIZATION: K-ADDITIVE FORM

Chevaleyre et al. [2] conjectured that ESWO$_{\texttt{2-additive}}$ is NP-complete. We solve this conjecture in the affirmative, and we prove that even ESWO$_{\texttt{1-additive}}$ is NP-complete. Furthermore, we show the same result for NPSWO$_{\texttt{1-additive}}$.

THEOREM 5.1. *For each $k \geq 1$,* ESWO$_{k\texttt{-additive}}$ *and* NPSWO$_{k\texttt{-additive}}$ *are* NP-*complete.*

Proof. That ESWO$_{k\texttt{-additive}}$ and NPSWO$_{k\texttt{-additive}}$ are in NP, for any fixed $k \geq 1$, is again easy to see.

To show that ESWO$_{k\texttt{-additive}}$ is NP-hard, we give a reduction from the well-known NP-complete problem PARTITION (see, e.g., Garey and Johnson [8]) to ESWO$_{\texttt{1-additive}}$.[7] PARTITION is defined as follows: Given a nonempty sequence $c_1, c_2, \ldots, c_s$ of positive integers such that $C = \sum_{i=1}^{s} c_i$ is even, is there a subset $I \subseteq S = \{1,2,\ldots,s\}$ such that $\sum_{i \in I} c_i = \sum_{i \in (S-I)} c_i$?

So, given an instance $(c_1, c_2, \ldots, c_s)$ of PARTITION, where $C = \sum_{i=1}^{s} c_i$ is even, we construct an instance (A,R,U,K) of ESWO$_{\texttt{1-additive}}$ as follows. There are two agents in $A = \{a_1, a_2\}$ and s resources in $R = \{r_1, r_2, \ldots, r_s\}$. (Recall that each resource can be held by one agent only, since resources are indivisible and nonshareable.) For $i \in \{1,2\}$, agent a_i's utilities are set to $u_i(\{r_j\}) = c_j$, $1 \leq j \leq s$, which means a_i's bid for the single resource r_j is c_j, and $u_i(\emptyset) = 0$. Finally, set $K = C/2$. Since egalitarian social welfare gives the utility of the agent that is worst off and since the sum of all utilities equals C, it follows that there exists an allocation $X \in \Pi_{2,s}$ such that $sw_e(X) \geq K$ (in fact, even $sw_e(X) = K$) if and only if there exists a partition.

The same reduction except with K chosen to be $(C/2)^2$ can be used for NPSWO$_{\texttt{1-additive}}$. If a partition exists, the product of the utilities both agents can realize in the corresponding allocation is exactly $(C/2)^2$, since the sum of all utilities equals C. Conversely, if there does not exist any partition, then for all allocations $X \in \Pi_{2,s}$ there is some $\lambda_X > 0$ such that one agent can realize a utility of $C/2 + \lambda_X$, whereas the other agent can realize only $C/2 - \lambda_X$ in X. Hence, the Nash product is

$$(C/2 + \lambda_X)(C/2 - \lambda_X) = (C/2)^2 - \lambda_X^2 < (C/2)^2,$$

which establishes NP-completeness of NPSWO$_{\texttt{1-additive}}$. ❑

Chevaleyre et al. [2] conjectured that XUSWO$_{\texttt{2-additive}}$ is DP-complete. Theorem 5.2 establishes not only this claim but also the corresponding result for exact egalitarian social welfare optimization.

THEOREM 5.2. *For each $k \geq 2$,* XUSWO$_{k\texttt{-additive}}$ *and* XESWO$_{k\texttt{-additive}}$ *are* DP-*complete.*

Proof. Membership of XUSWO$_{k\texttt{-additive}}$ in DP, for each fixed $k \geq 2$, can be seen as in the proof of Theorem 4.4.

To show DP-hardness of XUSWO$_{\texttt{2-additive}}$ (recall Footnote 7 for why it is enough to consider the case of $k = 2$ in both claims of Theorem 5.2), note that Wagner [16] proved DP-completeness of the exact version of INDEPENDENT SET, denoted EXACT INDEPENDENT SET (XIS, for short): Given an undirected graph G and a nonnegative number K, is it true that the size of a maximum independent set[8] of G is exactly K? Chevaleyre et al. [3] provided a reduction from INDEPENDENT SET (IS, for short) to USWO$_{\texttt{2-additive}}$ (which shows the NP-hardness of the latter problem). Their reduction satisfies that the size of a maximum independent set of the given graph equals the maximum utilitarian social welfare of the MARA setting constructed (maximized over all possible allocations). Combining these two results, we immediately obtain a reduction showing XIS $\leq_{\mathrm{m}}^{\mathrm{p}}$ XUSWO$_{\texttt{2-additive}}$, which establishes DP-hardness of XUSWO$_{\texttt{2-additive}}$.

In a bit more detail, as in [3] this reduction actually consists of two steps:[9]

1. from XIS to XUSWO$_{\texttt{3-additive}}$ (which already proves DP-hardness of XUSWO$_{k\texttt{-additive}}$ for each $k \geq 3$), and
2. from XUSWO$_{\texttt{3-additive}}$ to XUSWO$_{\texttt{2-additive}}$.

The latter reduction, XUSWO$_{\texttt{3-additive}} \leq_{\mathrm{m}}^{\mathrm{p}}$ XUSWO$_{\texttt{2-additive}}$, is possible since each instance of XUSWO$_{\texttt{3-additive}}$ can be transformed into an instance of XUSWO$_{\texttt{2-additive}}$ with the same utilitarian social welfare, where the size of the instance increases by only a linear factor (see Chevaleyre et al. [3] for details).

Now we prove that XESWO$_{k\texttt{-additive}}$ is also DP-complete for each fixed $k \geq 2$. Membership of XESWO$_{k\texttt{-additive}}$ in DP is easy to see.

Again, we make use of Lemma 4.3 to show hardness for DP. This time we apply the lemma with $A =$ CHROMATIC NUMBER (see, e.g., Garey and Johnson [8]) and $B =$ XESWO$_{\texttt{2-additive}}$. CHROMATIC NUMBER is known to be NP-complete and is defined as follows: Given a graph $G = (V,E)$ and a positive integer $k \leq \|V\|$, decide whether it is possible to color the vertices of G with at most k colors such that any two adjacent vertices have distinct colors. Such a coloring is said to be *legal*. We assume the vertices in V to be labeled from v_1 through $v_{\|V\|}$.

To apply Lemma 4.3, let $G = (V^G, E^G)$ and $H = (V^H, E^H)$ be two given graphs and k^G and k^H be two given positive integers

[7] Since for each $k > 1$, 1-additive utilities can be written as k-additive utilities (namely, by setting $u_i(T) = 0$ for all $T \subseteq R$ with $1 < \|T\| \leq k$, see Conitzer et al. [5]), ESWO$_{\texttt{1-additive}}$ is a restriction of ESWO$_{k\texttt{-additive}}$. Thus, proving ESWO$_{\texttt{1-additive}}$ NP-hard immediately yields NP-hardness of ESWO$_{k\texttt{-additive}}$ for all $k > 1$.

[8] An *independent set of a graph* G is a subset S of the vertex set of G such that no two vertices in S are adjacent.

[9] This approach of presenting two reductions is necessary because the value k in the k-additive representation form corresponds to the maximum vertex degree of the graph in the given XIS (respectively, IS) instance, and since XIS (respectively, IS) can be solved in polynomial time when this graph has a maximum vertex degree of at most two, i.e., the problem XIS restricted to graphs with maximum vertex degree at most two is not DP-complete and the thus restricted problem IS is not NP-complete.

such that if H is legally colorable with at most k^H colors then G is legally colorable with at most k^G colors.

First, we introduce the resources. We define one resource r_i^G, $1 \leq i \leq \|V^G\|$, for each vertex in G and one resource r_j^H, $1 \leq j \leq \|V^H\|$, for each vertex in H. Now, we form the agents. We define k^G agents a_i^G, $1 \leq i \leq k^G$, and k^H agents a_j^H, $1 \leq j \leq k^H$, to represent the colors. Furthermore, we need dummy agents $\overline{a}_i^G$ and $\overline{a}_j^H$, where $1 \leq i \leq \|V^G\|$ and $1 \leq j \leq \|V^H\|$. The utilities of the agents a_i^G, $\overline{a}_i^G$, a_j^H, and $\overline{a}_j^H$ now depend on the graphs G and H:

- Each agent a_i^G, $1 \leq i \leq \|V^G\|$, has a utility of two for each bundle containing a single resource r_s^G, $1 \leq s \leq \|V^G\|$, and a utility of $-2\|V^G\|$ for any bundle $\{r_m^G, r_n^G\}$ if and only if $\{v_m^G, v_n^G\}$ is an edge in E^G.
- Analogously, we define the utilities of the agents a_j^H, $1 \leq j \leq \|V^H\|$, by replacing G with H above.
- Each dummy agent $\overline{a}_i^G$, $1 \leq i \leq \|V^G\|$, and each dummy agent $\overline{a}_j^H$, $1 \leq j \leq \|V^H\|$, has a utility of two for the empty bundle of resources.
- Each dummy agent $\overline{a}_i^G$, $1 \leq i \leq \|V^G\|$, has a utility of -2 for the bundle containing only the single resource r_i^G.
- Each dummy agent $\overline{a}_j^H$, $1 \leq j \leq \|V^H\|$, has a utility of -1 for the bundle containing only the single resource r_j^H.
- To make sure the dummy agents can get only the resources corresponding to their names (when maximizing egalitarian social welfare), each $\overline{a}_i^G$ has a utility of -3 for any bundle containing only the single resource r_j^G, $i \neq j$, and each $\overline{a}_j^H$ has a utility of -3 for any bundle containing only r_i^H, $j \neq i$.
- To prevent any agent a_i^G or $\overline{a}_s^G$ from getting any resource r_m^H, we set the utility of a_i^G and $\overline{a}_s^G$ for each bundle T of resources, $1 \leq \|T\| \leq 2$, containing any r_m^H to be $-\|V^G\| \cdot \|V^H\|$.
- We do the same for any agent a_j^H and $\overline{a}_t^H$ and any bundle T, $1 \leq \|T\| \leq 2$, containing any resource r_n^G.
- For all agents, the utilities of all other bundles T of resources with $\|T\| \leq 2$ are set to zero.

Now we form our MARA setting (A,R,U) by

$$
\begin{aligned}
A &= \{a_i^G \mid 1 \leq i \leq k^G\} \cup \{a_j^H \mid 1 \leq j \leq k^H\} \cup \\
&\quad \{\overline{a}_s^G \mid 1 \leq s \leq \|V^G\|\} \cup \{\overline{a}_t^H \mid 1 \leq t \leq \|V^H\|\}, \\
R &= \{r_s^G \mid 1 \leq s \leq \|V^G\|\} \cup \{r_t^H \mid 1 \leq t \leq \|V^H\|\},
\end{aligned}
$$

and the related utilities U as described above. Finally, we choose the parameter for XESWO$_{2\text{-additive}}$ to be $K = 1$.

According to Lemma 4.3 consider the following three cases:

1. Suppose G is legally colorable with k^G colors and H is legally colorable with k^H colors. Without loss of generality, the resources corresponding to the vertices colored with color i, $1 \leq i \leq k^G$, can be given to agent a_i^G. Since those vertices colored with the same color are not adjacent, all agents can realize only positive utilities of at least two. The same holds for H and a_j^H. Since all resources are distributed among the agents a_i^G and a_j^H, each of the agents $\overline{a}_s^G$ and $\overline{a}_t^H$ can realize a utility of two for the empty bundle. Thus, each agent can realize a utility of at least two in this allocation and its egalitarian social welfare thus is greater than $K = 1$.
2. Suppose G is legally colorable with k^G colors but H is not legally colorable with k^H colors. Again, all agents associated with G can realize a utility of at least two. Since H is not legally colorable with k^H colors, there is at least one pair $\{v_m, v_n\}$ of adjacent vertices, which needs to be colored with the same color. To maximize egalitarian social welfare, it is not possible to give both resources to the same agent, because he or she has a utility of $-2\|V^H\|$ for owning both resources at the same time. This would lead to a utility of at most zero. So one of these resources has to be given either to dummy agent $\overline{a}_m^H$ or to dummy agent $\overline{a}_n^H$. But both these agents can realize a utility of exactly one, and thus the egalitarian social welfare in this allocation equals the parameter $K = 1$.
3. If G is not legally colorable with k^G colors, it does not matter whether H is legally colorable with k^H colors or not, since if G is not legally colorable with k^G colors then, analogously to the former case, there is an agent $\overline{a}_s^G$ who can realize only a utility of zero, so the egalitarian social welfare in the corresponding allocation is less than $K = 1$.

Applying Lemma 4.3, this proves the theorem. ❑

6. DISCUSSION OF INAPPROXIMABILITY OF SOCIAL WELFARE OPTIMIZATION

In the previous two sections, we have shown that the decision versions of certain social welfare optimization problems are intractable: either NP-complete or DP-complete. It is natural to ask whether the optimization problems associated with these decision problems are intractable as well, or whether they allow efficient approximation schemes. In this section, we briefly discuss inapproximability results for social welfare optimization, where we focus on the bundle form.

Indeed, one inapproximability result immediately follows from the work of Chevaleyre et al. [4], who give a reduction from SET PACKING (see, e.g., Garey and Johnson [8]) to USWO$_{\texttt{bundle}}$ to show NP-hardness of the latter problem. Given a collection $\mathscr{C}$ of finite, nonempty subsets of some base set S, a *set packing for* $\mathscr{C}$ is a collection $\mathscr{C}' \subseteq \mathscr{C}$ that contains only pairwise disjoint sets from $\mathscr{C}$. The optimization problem associated with SET PACKING is known as MAXIMUM SET PACKING and is defined as follows. (Note that we present optimization problems in a different format than decision problems by changing the "Question" field into a "Task" field, where the task is to find some optimum value.)

MAXIMUM SET PACKING	
Given:	A set S and a collection $\mathscr{C}$ of finite, nonempty subsets of S.
Task:	Find the cardinality of a maximum-size set packing for $\mathscr{C}$.

The best approximation results known for MAXIMUM SET PACKING are due to Halldórsson et al. [9], who show that this problem can be approximated in, respectively, $\mathscr{O}(\sqrt{\|S\|})$ and $\mathscr{O}(\|\mathscr{C}\|/\log^2\|\mathscr{C}\|)$. They also note that MAXIMUM SET PACKING and MAXIMUM INDEPENDENT SET are mutually reducible by approximation factor preserving reductions (see Vazirani [15] for a formal definition),[10] and the same is true for MAXIMUM

[10] Informally speaking, this means that there is a one-to-one correspondence between the independent sets in the graph and the packings of the set system, so this reduction together with an α-approximation for MAXIMUM INDEPENDENT SET yields an α-approximation for MAXIMUM SET PACKING, and vice versa.

SET PACKING and MAXIMUM CLIQUE. Håstad [10] proved that MAXIMUM CLIQUE cannot be approximated within a factor of $n^{1-\varepsilon}$ unless NP = ZPP, where n is the number of vertices in the given graph, ε is an arbitrarily small positive constant, and ZPP is the complexity class "zero-error probabilistic polynomial time." Thus the same inapproximability holds for MAXIMUM SET PACKING, where $n = \|\mathscr{C}\|$.

The *optimization problem* associated with $\text{USWO}_{\texttt{bundle}}$ asks for the maximum utilitarian social welfare that can be reached in any allocation (compare this problem with $\text{XUSWO}_{\texttt{bundle}}$):

MAXIMUM UTILITARIAN SOCIAL WELFARE$_{\texttt{bundle}}$	
Given:	A MARA setting (A,R,U), where $\|A\| = \|U\| = n$ and $\|R\| = m$ and where `form` indicates how the utility functions in U are represented.
Task:	Determine $\max\{sw_u(X) \mid X \in \Pi_{n,m}\}$.

We use the shorthand $\text{MAXUSW}_{\texttt{bundle}}$ to denote the above problem, and we let $\text{MAXESW}_{\texttt{bundle}}$ and $\text{MAXNPSW}_{\texttt{bundle}}$ denote the corresponding optimization problems for egalitarian social welfare and for the Nash product.

Now, a close inspection of the above-mentioned reduction of Chevaleyre et al. [4] from SET PACKING to $\text{USWO}_{\texttt{bundle}}$ reveals that it can be viewed as a reduction from MAXIMUM SET PACKING to $\text{MAXUSW}_{\texttt{bundle}}$ that is also approximation factor preserving. Thus the inapproximability of MAXIMUM SET PACKING transfers to $\text{MAXUSW}_{\texttt{bundle}}$ as well.

It is an interesting open question whether one can prove similar approximation factor preserving reductions from some hard-to-approximate optimization problem to the optimization problems $\text{MAXESW}_{\texttt{bundle}}$ or $\text{MAXNPSW}_{\texttt{bundle}}$. Our reduction showing that, e.g., $\text{ESWO}_{\texttt{bundle}}$ is NP-complete (see Theorem 4.1) is not suitable for this purpose. Even if it were reducing from an NP problem whose optimization version is provably hard to approximate unless NP = ZPP (such as MAXIMUM SET PACKING), this reduction would not be approximation factor preserving because it allows only two possible utilities for the agents, zero and one, and the def nition of egalitarian social welfare implies that the maximum social welfare is either zero or one as well. We feel that this obstacle is inherent to the problem $\text{MAXESW}_{\texttt{bundle}}$: While the task for $\text{MAXUSW}_{\texttt{bundle}}$ is to maximize the sum of the agents' individual utilities, the task for $\text{MAXESW}_{\texttt{bundle}}$ is to maximize the minimum utility any agent can realize. For the same reason, our reduction for $\text{NPSWO}_{\texttt{bundle}}$ is not suitable to show that $\text{MAXNPSW}_{\texttt{bundle}}$ is hard to approximate.

7. CONCLUSION AND FUTURE WORK

We conclude this paper by mentioning some interesting open issues. For example, the question of whether the complexity results shown here also hold for the SLP form remains open; with respect to the SLP form, only utilitarian social welfare optimization is known to be NP-complete [7]. Another interesting open question regards the complexity of *exact* social welfare optimization with respect to the Nash product in either representation form. We conjecture that DP-completeness holds in each case.

Regarding approximation algorithms, some open questions have been mentioned in Section 6 already: Study the (in)approximability of $\text{MAXESW}_{\texttt{bundle}}$ and $\text{MAXNPSW}_{\texttt{bundle}}$. The same question applies to other representation forms, such as the k-additive form.

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